Quadrature of Quadratures: Compressed Sampling by Caratheodory-Tchakaloff Points

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A discrete version of Tchakaloff theorem on the existence of positive algebraic cubature formulas (via Caratheodory theorem on conical vector combinations), entails that the information required for multivariate polynomial approximation can be suitably compressed.

The framework here is approximating a discrete measure by another one, with the same polynomial moments up to a certain degree, and a (much) smaller support.

Extracting such "Caratheodory-Tchakaloff points" from the support of discrete measures by Linear or Quadratic Programming, we obtain compression of Algebraic Quadrature and Least Squares approximation on multivariate compact sets and manifolds.

Since the $\ell^1$-norm of the weights is constant by construction, this technique differs substantially from popular compressed sensing methods based on $\ell^1$-minimization.

Applications arise, for example, in Geospatial Analysis (construction of efficient sensor networks), Optical System Design (ray tracing on variable geometry pupil masking), Numerical PDEs (efficient quadrature formulas for polygonal/polyhedral finite elements), Polynomial Optimization (discretization of Lasserre measure-based hierarchies).

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References


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Abstract
By a discrete version of Tchakaloff Theorem on positive quadrature (via Caratheodory Theorem on conical linear combinations of points) the Caratheodory-Tchakaloff (C-T) compressed sampling sets for multivariate polynomial quadrature, least squares and optimization are presented.

Discrete Tchakaloff theorem

THEOREM 1 Let be a positive measure with compact support in and be the space of real d-variate polynomials of total degree , restricted to , then there are such that points , , , and positive real numbers such that
\[
\int f(x)\,d\mu = \sum_{j=1}^{d+1} w_j f(t_j), \quad \forall f \in P^d.
\]

Algebraic quadrature for nonstandard regions

\[
\int f(x)\,d\mu = \sum_{j=1}^{d+1} w_j f(t_j), \quad \forall f \in P^d
\]

Compressed quadrature

\[
V = \{\mu \geq 0, \mu \in \mathbb{R}^d, \mu \neq 0\}
\]

Implementation by NonNegative Least Squares

let be a discrete measure with finite support , and masses (weights) \(\lambda_i\)

\[
\int f(x)\,d\mu = \sum_{i=1}^{N} \lambda_i f(x_i)
\]

Least squares example

\[
\left| V^* - b \right| = \min \| V \|_2 - \left| b \right|_2
\]

Quadrature-based Polynomial Optimization

\[
A = \left\{ a \in \mathbb{R}^d : \mu(a) = \frac{1}{d} \sum_{i=1}^{d} g_i(a_i) \right\}
\]

References

[4] D. de Klerk, M. Laurent and Z. Sun, Convergence analysis for Lasserre’s measure-based hier-

Figure 2: CR point set for CR = 10, on a smooth function.

Table 2: CR, moment residual and RMSSE for the Gaussian f(x) at (x1) = supd(μ) and the present function f(x) = (x1) + (x2) + (x3) + (x4) + (x5).

Table 3: Compression of multivariate discrete measures and applications to near optimality of polynomial meshes (sequence of finite norming subsets) and compression of point sets (smooth convex functions).

Table 4: Compression of multivariate discrete measures and applications to near optimality of polynomial meshes (sequence of finite norming subsets) and compression of point sets (smooth convex functions).

Table 5: Compression of multivariate discrete measures and applications to near optimality of polynomial meshes (sequence of finite norming subsets) and compression of point sets (smooth convex functions).